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THE STABILITY OF THE EQUILIBRIUM OF A PENDULUM FOR VERTICAL OSCILLATIONS OF THE POINT OF SUSPENSION†

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The motion of a pendulum, the point of suspension of which is subject to vertical harmonic oscillations of arbitrary frequency and amplitude, is considered. A complete rigorous solution of the non-linear problem of the stability of the relative positions of equilibrium of the pendulum along the vertical is given.

Considerable attention has been given to the problem of the stability of the equilibrium of a pendulum when its point of suspension is subjected to harmonic oscillations. However, this problem has only been solved completely using the linearized equations of perturbed motion (see, for example, [1–3]). A rigorous solution of the problem of the stability of the normal position of equilibrium of the pendulum has been given in [4], but only for small amplitudes of oscillation of the point of suspension. A rigorous sufficient condition for the instability of the inverted position of the pendulum was obtained in [5]. A non-rigorous analysis of the non-linear problem of the stability of an inverted pendulum was carried out in [6, 7].

1. FORMULATION OF THE PROBLEM

Suppose the pendulum is regarded as an absolutely solid weightless rod of length l, rotating around one of its ends and having a point mass m at its other end. All the results obtained below can easily be extended to the case of a physical pendulum; only the value of l needs to be changed to its reduced length, equal to r^2d^{-1}, where r is the radius of inertia and d is the distance from the centre of gravity to the point of suspension. The point of suspension 0 of the pendulum is subjected to harmonic oscillations along the vertical of amplitude a and frequency ω: $z_0 = a\cos\omega t$, where z_0 is the displacement of the point of suspension from a certain fixed position 0 (Fig. 1).

The equation of motion has the form

$$\ddot{\varphi}+(\alpha+\beta\cos\tau)\sin\varphi=0,\quad \alpha=g/(\omega^2 l),\quad \beta=a/l \tag{1.1}$$

where φ is the angle of deflection of the pendulum from the vertical, and the dots denote differentiation with respect to dimensionless time $\tau = \omega t$.

Equation (1.1) has particular solutions $\varphi = 0$ and $\varphi = \pi$, corresponding to positions of relative equilibrium of the pendulum along the vertical. When $\varphi = 0$ the point of suspension lies above the centre of gravity, and when $\varphi = \pi$ it lies below the centre of gravity. Following [8], we will call the first position the normal position and the second position the inverted position of the pendulum.

The purpose of the present paper is to obtain a rigorous solution of the problem of the Lyapunov stability of these positions of the pendulum for all possible values of the parameters α and β.

Assuming $\varphi = q$, $\dot{\varphi} = p$, the equations of the perturbed motion for the case of the normal position of the pendulum can be written in the Hamiltonian form

$$dq/d\tau = \partial H/\partial p,\; dp/d\tau = -\partial H/\partial q \tag{1.2}$$

$$H = \tfrac{1}{2}p^2-(\alpha+\beta\cos\tau)\cos q \tag{1.3}$$

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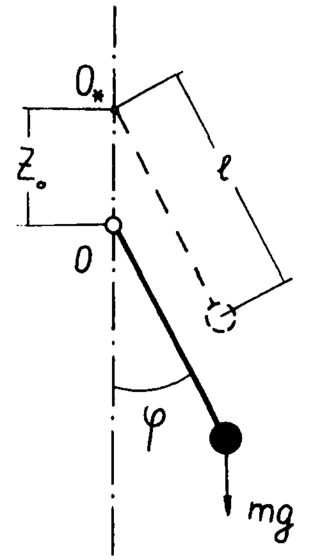

Fig. 1.

Similarly, putting $\varphi = \pi + q$, $\dot\varphi = p$, we obtain the Hamiltonians of the equation of perturbed motion for the case of the inverted pendulum. The Hamiltonian corresponding to it reduces to Hamiltonian (1.3) when τ is replaced by $\tau + \pi$ and α is replaced by $-\alpha$.

Hence, when solving the problem of the stability of the positions of relative equilibrium of the pendulum along the vertical we will consider the stability of the solution $q = p = 0$ of Eqs (1.2), assuming that the parameters α and β take any values from the half-plane $-\infty < \alpha < \infty$, $\beta \geqslant 0$ in the Hamilton function (1.3). As a result of an investigation this half-plane can be split into regions of stability and instability. Those of these were $\alpha \geqslant 0$, $\beta \geqslant 0$ will be regions of stability and instability of the normal position of equilibrium of the pendulum. The regions in which $\alpha \leqslant 0$, $\beta \geqslant 0$, after mirror reflection in the axis $\alpha = 0$, give the regions of stability and instability of the inverted position of the pendulum.

2. THE LINEAR EQUATIONS OF PERTURBED MOTION AND THE STABILITY IN THE FIRST APPROXIMATION

In the neighbourhood of the point $q = p = 0$ the Hamilton function (1.3) can be represented in the form of a converging series in powers of q and p

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}(\alpha + \beta\cos\tau)q^2 - 1/24(\alpha + \beta\cos\tau)q^4 + \ldots \tag{2.1}$$

The term in (2.1) which is independent of q and p is omitted.

The system of equations (1.2), linearized in the neighbourhood of the point $q = p = 0$, is equivalent to the Mathieu equation

$$\ddot{q} + (\alpha + \beta\cos\tau)q = 0 \tag{2.2}$$

There is an extensive literature devoted to investigating this equation. The results and a fairly complete bibliography can be found in [1–3]. We will briefly present some information which will be needed below.

Suppose $\mathbf{X}(\tau)$ is the fundamental matrix of the solutions of the linearized system (1.2) which satisfies the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the second-order unit matrix. The elements $x_{11}(\tau)$ and $x_{12}(\tau)$ of the first row of this matrix satisfy Eq. (2.2), while the elements of the second row are obtained from them by differentiation with respect to τ: $x_{21} = \dot{x}_{11}$, $x_{22} = \dot{x}_{12}$. The diagonal elements x_{11} and x_{12} are even functions of τ, while x_{12} and x_{21} are odd functions of τ.

In the characteristic equation of the linearized system (1.2)

$$\rho^2 - 2A\rho + 1 = 0 \tag{2.3}$$

we have $A = x_{11}(2\pi) = x_{22}(2\pi)$.

In Fig. 2 we show regions of stability and instability of Eq. (2.2) in the half-plane $-\infty < \alpha < \infty$, $\beta \geqslant 0$. The regions of instability (regions of parametric resonance) are shown hatched. In these regions the modulus of one of the roots of Eq. (2.3) is greater than unity. Consequently, by Lyapunov's theorem on stability in the first approximation [9], instability occurs here not only for the linear equation (2.2) but also for the complete non-linear system of equations of perturbed motion (1.2). In the non-hatched part of Fig. 2 the conditions of stability are satisfied in the linear approximation. Here the roots of Eq. (2.3) are complex-conjugate and have moduli equal to unity. Then

$$A = \cos 2\pi\lambda, \quad x_{12}(2\pi)x_{21}(2\pi) = -\sin^2 2\pi\lambda < 0$$

where $\pm i\lambda$ are characteristic indices (i is the square root of -1 and $\lambda > 0$).

The sets of region of instability and regions of stability in the linear approximation are denumerable. We will denote by g_n ($n = 1, 2, \ldots$) the region of stability which, when $\beta \to 0$, transfers into the interval $(n-1)^2/4 < \alpha < n^2/4$ of the $\beta = 0$ axis. We will denote the curvilinear boundaries of the regions g_{2m-1} and g_{2m} ($m = 1, 2, \ldots$) by $\gamma_c^{(2m-2)}$, $\gamma_c^{(2m-1)}$ and $\gamma_s^{(2m-1)}$, $\gamma_s^{(2m)}$, respectively. The curves of $\gamma_c^{(k)}$ and $\gamma_s^{(k)}$ intersect on the $\beta = 0$ axis at the points $\alpha = k^2/4$ ($k = 1, 2, \ldots$), from which, for small β, regions of parametric resonance are produced. The boundary curves $\gamma_c^{(k)}$ and $\gamma_s^{(k)}$ of these regions have a tangency of the order of $k-1$ ($k = 1, 2, \ldots$) as $\beta \to 0$.

All the boundary curves intersect the $\alpha = 0$ axis and do not terminate in a finite region. For fixed β the regions of stability are wider the larger the value of α. For large values of β the regions of stability become very narrow and approach the curves for which the slope of the tangent is equal to -1.

For values of the parameters α and β belonging to the boundary curves the roots of Eq. (2.3) are equal. On the $\gamma_c^{(0)}$, $\gamma_c^{(2k)}$, $\gamma_s^{(2k)}$ ($k = 1, 2, \ldots$) curves we have first-order resonance ($\rho_1 = \rho_2 = 1$), while on curves $\gamma_c^{(2k-1)}$, $\gamma_s^{(2k-1)}$ ($k = 1, 2, \ldots$) we have second-order resonance ($\rho_1 = \rho_2 = -1$). Here the elementary dividers of the matrix $\mathbf{X}(2\pi) - \rho\mathbf{E}$ are non-prime for $\beta > 0$.

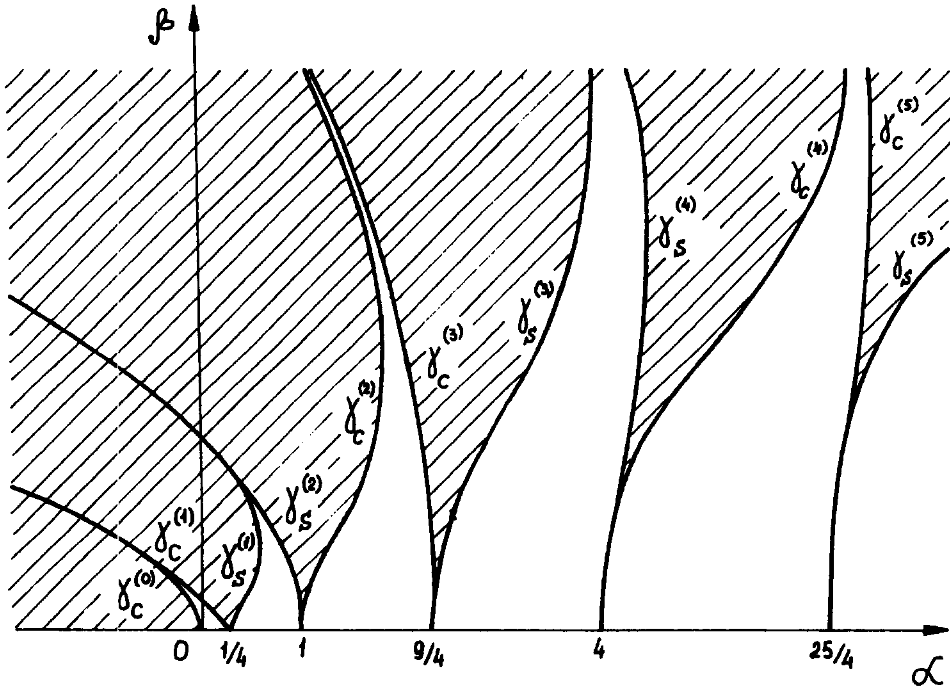

Fig. 1.

We will denote the linearly independent solutions of Mathieu's equation (2.2) on the $\gamma_c^{(m)}$ ($m = 0, 1, 2, \ldots$) curves by $\varphi_1^{(m)}$ and $\varphi_2^{(m)}$, and on the $\gamma_s^{(m)}$ ($m = 1, 2, \ldots$) curves by $\psi_1^{(m)}$ and $\psi_2^{(m)}$. Then

$$\varphi_1^{(m)}(\tau) = \mathrm{ce}_m(\tau/2, -2\beta),\quad \varphi_2^{(m)}(\tau) = \mathrm{fe}_m(\tau/2, -2\beta),$$
$$\psi_1^{(m)}(\tau) = \mathrm{se}_m(\tau/2, -2\beta),\quad \psi_2^{(m)}(\tau) = \mathrm{ge}_m(\tau/2, -2\beta)$$

Here ce_m and se_m are even and odd Mathieu functions of the first kind, while fe_m and ge_m are the corresponding Mathieu functions of the second kind. The functions $\varphi_1^{(m)}$ and $\psi_1^{(m)}$ are 2π-periodic in τ for even m and 4π-periodic when m is odd. The functions $\varphi_2^{(m)}$ and $\psi_2^{(m)}$ are non-periodic and unbounded, and approach infinity as the first power of τ. Consequently, for values of the parameters α and β belonging to the boundary curves there will be instability in the linear approximation.

A rigorous solution of the problem of stability inside regions g_n of stability in the linear approximation and on the boundary curves requires a consideration of the complete non-linear system of equations of perturbed motion (1.2). The corresponding necessary methods and algorithms have already been developed (see [10–13]).

We will prove the following assertion in Sections 3–5.

Theorem. For values of the parameters α and β lying inside the regions of stability in the linear approximation or on the boundaries of $\gamma_c^{(2k-1)}, \gamma_s^{(2k)}$ ($k = 1, 2, \ldots$), the solution $q = p = 0$ of system (1.2) is Lyapunov stable, and on the boundaries of $\gamma_c^{(0)}, \gamma_c^{(2k)}, \gamma_s^{(2k-1)}$ ($k = 1, 2, \ldots$) we have instability.

3. NORMALIZATION OF THE LINEARIZED EQUATIONS OF PERTURBED MOTION

According to [10–13], for a rigorous solution of the problem of stability we must first reduce to normal form the part of the Hamiltonian of the perturbed motion (2.1) that is quadratic in q and p.

We will first consider the region g_n of stability in the linear approximation. In this case, by a linear real 2π-periodic in τ canonical change of variables $q, p \to q_*, p_*$

$$q = n_{11}(\tau)q_* + n_{12}(\tau)p_*,\quad p = n_{21}(\tau)q_* + n_{22}(\tau)p_* \tag{3.1}$$

the Hamiltonian (2.1) is reduced to the form

$$H = \tfrac{1}{2}\lambda(q_*^2 + p_*^2) - 1/24(\alpha + \beta\cos\tau)(n_{11}q_* + n_{12}p_*)^4 + O_6 \tag{3.2}$$

where O_6 is the set of terms of the sixth and higher powers in q_*, p_*.

The quantity λ in (3.2) is defined non-uniquely by the relation $\cos 2\pi\lambda = A$. The non-uniqueness can be eliminated if we use the continuity of the characteristic indices in β by noting that when $\beta = 0$ we have $\lambda = \alpha^{1/2}$. We obtain that

$$\lambda = \begin{cases} (2\pi)^{-1}\arccos A + n - 1 & \text{for } g_{2n-1}\ (n = 1, 2, \ldots) \\ -(2\pi)^{-1}\arccos A + n & \text{for } g_{2n} \end{cases}$$

In the change of variables (3.1) we have

$$\begin{gathered} n_{i1} = \kappa^{-1/2}(\mu_i\cos\lambda\tau + \nu_i\sin\lambda\tau),\quad n_{i2} = \kappa^{-1/2}(-\mu_i\sin\lambda\tau + \nu_i\cos\lambda\tau) \\ \kappa = x_{12}(2\pi)\sin 2\pi\lambda > 0 \\ \mu_i = \sin 2\pi\lambda x_{i2}(\tau),\quad \nu_i = -x_{12}(2\pi)x_{i1}(\tau)(i = 1, 2) \end{gathered} \tag{3.3}$$

Suppose now that the parameters α and β belong to the boundary curves. As was pointed out above, on these curves one obtains first- or second-order resonance, while the elementary divisors of the matrix $\mathbf{X}(2\pi) - \rho\mathbf{E}$ are non-simple. For first-order resonance there is a linear, real, canonical, 2π-periodic change of variables $q, p \to q_*, p_*$ which reduces the Hamiltonian (2.1) to the form

$$H=\frac{1}{2}\delta p_*^2-\frac{1}{24}(\alpha+\beta\cos\tau)(n_{11}q_*+n_{12}p_*)^4+O_6 \tag{3.4}$$

where the quantity δ is equal to 1 or -1, and its specific value is determined during linear normalization. The normalizing change of variables

$$\|qp\|'=\mathbf{N}(\tau)\|q_*p_*\|' \tag{3.5}$$

is specified by the simplectic matrix $\mathbf{N}=\|n_{ij}\|$ of the form

$$\mathbf{N}=\mathbf{X}(\tau)\mathbf{P}\mathbf{Q}(\tau) \tag{3.6}$$

where

$$\mathbf{Q}=\begin{Vmatrix}1 & -\delta\tau\\ 0 & 1\end{Vmatrix}$$

and the matrix $\mathbf{P}$ and the quantity δ are defined as follows. If $x_{12}(2\pi)\neq 0$ then $\delta=\operatorname{sign} x_{12}(2\pi)$, and

$$\mathbf{P}=\begin{Vmatrix}b & 0\\ 0 & b^{-1}\end{Vmatrix},\quad b=\left(|x_{12}(2\pi)|/(2\pi)\right)^{1/2}$$

If $x_{21}(2\pi)\neq 0$, then $\delta=-\operatorname{sign} x_{21}(2\pi)$, and

$$\mathbf{P}=\begin{Vmatrix}0 & c^{-1}\\ -c & 0\end{Vmatrix},\quad c=\left(|x_{21}(2\pi)|/(2\pi)\right)^{1/2}$$

Note that $x_{12}(2\pi)\,x_{21}(2\pi)=0$, but simultaneously quantities $x_{12}(2\pi)$ and $x_{21}(2\pi)$ cannot be zero since the elementary divisors of the matrix $\mathbf{X}(2\pi)-\mathbf{E}$ are non-prime.

For second-order resonance the Hamilton function (2.1) is reduced to the form (3.4) using the 4π-periodic in τ change (3.5) with the matrix $\mathbf{N}$ of the form (3.6). When determining the constant δ and the matrix $\mathbf{P}$ in the corresponding formulae, 2π must be replaced by 4π.

Later, in Section 5 when performing a non-linear analysis of the problem of stability on the boundary curves, the quantity δ and the element $n_{11}(\tau)$ of the matrix $\mathbf{N}$ will be required. We will consider four possible cases.

1. The curves $\gamma_c^{(2k)}$ ($k=0,1,2,\ldots$) of first-order resonances. These curves are the right boundaries of the regions of parametric resonance which, for small β, starting from the points $\alpha=k^2$ of the $\beta=0$ axis. On these we have

$$\mathbf{X}(\tau)=\begin{Vmatrix}\dfrac{\varphi_1^{(2k)}(\tau)}{\varphi_1^{(2k)}(0)} & \dfrac{\varphi_2^{(2k)}(\tau)}{\dot\varphi_2^{(2k)}(0)}\\[2mm] \dfrac{\dot\varphi_1^{(2k)}(\tau)}{\varphi_1^{(2k)}(0)} & \dfrac{\dot\varphi_2^{(2k)}(\tau)}{\dot\varphi_2^{(2k)}(0)}\end{Vmatrix} \tag{3.7}$$

$$x_{21}(2\pi)=0,\quad x_{12}(2\pi)=\frac{\varphi_2^{(2k)}(2\pi)}{\dot\varphi_2^{(2k)}(0)},\quad \delta=\operatorname{sign}x_{12}(2\pi) \tag{3.8}$$

$$n_{11}(\tau)=bx_{11}(\tau) \tag{3.9}$$

2. The curves $\gamma_s^{(2k)}$ ($k=1,2,\ldots$) of first-order resonances. These curves are the left boundaries of the regions of parametric resonance starting from the points $\alpha=k^2$ of the $\beta=0$ axis. Here

$$\mathbf{X}(\tau)=\begin{Vmatrix}\dfrac{\psi_2^{(2k)}(\tau)}{\psi_2^{(2k)}(0)} & \dfrac{\psi_1^{(2k)}(\tau)}{\dot\psi_1^{(2k)}(0)}\\ \dfrac{\dot\psi_2^{(2k)}(\tau)}{\psi_2^{(2k)}(0)} & \dfrac{\dot\psi_1^{(2k)}(\tau)}{\dot\psi_1^{(2k)}(0)}\end{Vmatrix} \tag{3.10}$$

$$x_{12}(2\pi)=0,\ \ x_{21}(2\pi)=\frac{\dot\psi_2^{(2k)}(2\pi)}{\psi_2^{(2k)}(0)},\ \ \delta=-\operatorname{sign}x_{21}(2\pi) \tag{3.11}$$

$$n_{11}(\tau)=-c\ \ x_{12}(\tau) \tag{3.12}$$

3. The curves $\gamma_s^{(2k-1)}$ ($k = 1, 2, \ldots$) of second-order resonances. These curves are the right boundaries of the regions of parametric resonance starting from the points $\alpha = (2k-1)^2/4$ of the $\beta = 0$ axis. On these curves the matrix $\mathbf{X}(t)$ and the quantities $x_{12}(4\pi)$, $x_{21}(4\pi)$, δ, $n_{11}(\tau)$ are specified by Eqs (3.10)–(3.12), in which the functions $\psi_i^{(2k)}$ ($i = 1, 2$) the superscript $2k$ must be replaced by $2k-1$, and 2π must be replaced by 4π.

4. The curves $\gamma_c^{(2k-1)}$ ($k = 1, 2, \ldots$) of second-order resonances. These are the left boundaries of the regions of parametric resonance starting from the points $\alpha = (2k-1)^2/4$ of the $\beta = 0$ axis. Here the matrix $\mathbf{X}(\tau)$ and the quantities $x_{12}(4\pi)$, $x_{21}(4\pi)$, δ, $n_{11}(\tau)$ are defined by (3.7)–(3.9) if in the functions $\varphi_i^{(2k)}$ ($i = 1, 2$) the superscript $2k$ is replaced by $2k-1$ and 2π is replaced by 4π.

The quantities $x_{12}(4\pi), x_{12}(4\pi)$ in cases 1 and 4 and $x_{21}(4\pi), x_{21}(4\pi)$ in cases 2 and 3 when $\beta > 0$ retain their sign over the whole corresponding boundary curve. Hence, to obtain the quantity δ it is sufficient to investigate the sign of these quantities for small β. By using the necessary expansions of the Mathieu functions from [2], we obtain that in cases 1–4 considered above when $0 < \beta \ll 1$ we have the following relations

$$1^\circ.\ \ x_{12}(2\pi)\sim\frac{\pi\beta^{2k}}{2^{2k-3}[(2k)!]^2}>0,\ \ 2^\circ.\ \ x_{21}(2\pi)\sim\frac{\pi\beta^{2k}}{2^{2k-1}[(2k-1)!]^2}>0$$

$$3^\circ.\ \ x_{21}(4\pi)\sim-\frac{\pi\beta^{2k-1}}{2^{2k-3}[2k-2)!]^2}<0,\ \ 4^\circ.\ \ x_{12}(4\pi)\sim-\frac{\pi\beta^{2k-1}}{2^{2k-5}[(2k-1)!]^2}<0$$

Consequently, in cases 1 and 3 the quantity δ is equal to 1, while in cases 2 and 4 we have $\delta = -1$.

4. NON-LINEAR ANALYSIS OF STABILITY IN THE REGIONS OF STABILITY IN THE LINEAR APPROXIMATION

In each of the regions of stability in the linear approximation there is one curve on which fourth-order resonance occurs (4λ is an integer). In the region g_n on this curve we have $4\lambda = 2n-1$, and for small β the curve issues from the point $\alpha = (2n-1)^2/16$ of the $\beta = 0$ axis and extends without limit in the direction of increasing values of β. Fourth-order resonance curves are not shown in Fig. 2.

Changing to a non-linear analysis of the stability of the solution $q = p = 0$ of system (1.2), we will first consider the non-resonance case when the parameters α and β lie inside the regions g_n of stability in the linear approximation without falling on the curves $4\lambda = 2n-1$ ($n = 1, 2, \ldots$). Then, by a close to identical, real, 2π-periodic in τ, analytic in x and y canonical change of variables of the Birkhoff transformation type, the Hamiltonian (3.2) can be reduced to the form

$$H=\tfrac{1}{2}\lambda(x^2+y^2)+\tfrac{1}{4}c_2(x^2+y^2)^2+O_6 \tag{4.1}$$

where c_2 is a constant quantity. If $c_2 \neq 0$, we have stability [10, 11].

According to [12], for the Hamilton function (3.2) we have

$$c_2=-\frac{1}{32\pi}\int_0^{2\pi}(\alpha+\beta\cos\tau)(n_{11}^2+n_{12}^2)^2\,d\tau$$

Substituting the functions n_{11} and n_{12} from (3.3) here and using the fact that the functions $\mu_1(\tau)$, $\nu_1(\tau)$ are solutions of the Mathieu equation (2.2), we obtain, after some reduction, the following expression for c_2

$$c_2 = -\frac{1}{32\pi\kappa^2}\int_0^{2\pi}\left[(\dot{\mu}_1 \nu_1 + \mu_1 \dot{\nu}_1)^2 + (\mu_1 \dot{\mu}_1 - \nu_1 \dot{\nu}_1)^2 + 2(\mu_1 \dot{\mu}_1 + \nu_1 \dot{\nu}_1)^2\right]d\tau < 0 \tag{4.2}$$

Since $c_2 \neq 0$, inside all the regions g_n ($n = 1, 2, \ldots$) when there are no fourth-order resonances we have stability.

Suppose the parameters α and β belong to fourth-order resonance curves. In this case the non-linear normalizing change of variables $q_*, p_* \to x, y$ reduces the Hamiltonian (3.2) to the form

$$H = \tfrac{1}{2}\lambda(x^2 + y^2) + \tfrac{1}{4}c_2(x^2 + y^2)^2 + (x_{40}\cos 4\lambda\tau - y_{40}\sin 4\lambda\tau)(x^4 -$$
$$-6x^2y^2 + y^4) - 4(y_{40}\cos 4\lambda\tau + x_{40}\sin 4\lambda\tau)xy(x^2 - y^2) + O_6 \tag{4.3}$$

In (4.3) the coefficient c_2 is the same as in (4.1) and the quantities x_{40} and y_{40} are constant. If $|c_2| > 4(x_{40}^2 + y_{40}^2)^{1/2}$, the solution $q = p = 0$ of system (1.2) is stable, and if we have the opposite sign in the latter inequality, we have instability [12].

According to [12] for the Hamiltonian (3.2) we have

$$x_{40} = \int_0^{2\pi}(\chi_1 \cos 4\lambda\tau + \chi_2 \sin 4\lambda\tau)d\tau, \quad y_{40} = \int_0^{2\pi}(-\chi_1 \sin 4\lambda\tau + \chi_2 \cos 4\lambda\tau)d\tau \tag{4.4}$$

$$\chi_1 = -\frac{1}{384\pi}(\alpha + \beta\cos\tau)[(n_{11}^2 + n_{12}^2)^2 - 8n_{11}^2 n_{12}^2]$$

$$\chi_2 = \frac{1}{96\pi}(\alpha + \beta\cos\tau)n_{11}n_{12}(n_{11}^2 - n_{12}^2)$$

Using expressions (3.3) for n_{11} and n_{12} we can convert expressions (4.4) to the form

$$x_{40} = \frac{1}{384\pi\kappa^2}\int_0^{2\pi}[4\mu_1^2\nu_1^2 - (\mu_1^2 - \nu_1^2)^2](\alpha + \beta\cos\tau)d\tau$$

$$y_{40} = \frac{1}{96\pi\kappa^2}\int_0^{2\pi}\mu_1\nu_1(\mu_1^2 - \nu_1^2)(\alpha + \beta\cos\tau)d\tau$$

In view of the fact that the integrand in the second of these equations is odd we have $y_{40} = 0$. The expression for x_{40} can be converted if we use the fact that the functions $\mu_1(\tau)$ and $\nu_1(\tau)$ are solutions of the Mathieu equation (2.2). We obtain

$$x_{40} = \frac{1}{128\pi\kappa^2}\int_0^{2\pi}\left[(\dot{\mu}_1\nu_1 + \mu_1\dot{\nu}_1)^2 - (\mu_1\dot{\mu}_1 - \nu_1\dot{\nu}_1)^2\right]d\tau \tag{4.5}$$

When the equality $y_{40} = 0$ is taken into account the condition for stability can be written in the form of the inequality $|c_2| > 4|x_{40}|$. It follows from (4.2) and (4.5) that this condition is satisfied on all the fourth-order resonance curves.

5. INVESTIGATION OF STABILITY FOR VALUES OF THE PARAMETERS BELONGING TO THE BOUNDARY CURVES

For boundary values of the parameters α and β, by using the non-linear normalizing transformation $q_*, p_* \to x, y$, the Hamilton function (3.4) can be reduced to the form [13]

$$H = \tfrac{1}{2}\delta y^2 + a_4 x^4 + O_6 \tag{5.1}$$

where a_4 is a constant quantity. If $a_4\delta > 0$, the solution $q = p = 0$ of system (1.2) is stable, while if $a_4\delta < 0$, we have instability [13].

The normalizing canonical transformation of Hamiltonian (3.4) to the form (5.1) has a period of $2\pi s$ with respect to τ, where $s = 1$ in cases 1 and 2 and $s = 2$ in cases 3 and 4. The coefficient a_4 in (5.1) can be calculated from the formula

$$a_4 = -\frac{1}{48\pi s}\int_0^{2\pi s}(\alpha+\beta\cos\tau)n_{11}^4 d\tau \tag{5.2}$$

The function $n_{11}(\tau)$ occurring here is defined by (3.9) and (3.12) in cases 1 and 2, and by similar equalities (see Section 3) in cases 3 and 4. Noting that this function is a solution of Mathieu equation (2.2), expression (5.2) can be converted to the form

$$a_4 = -\frac{1}{16\pi s}\int_0^{2\pi s} n_{11}^2 \dot{n}_{11}^2 d\tau$$

Hence it follows that in all possible cases of the boundary curves considered in Section 3 the quantity a_4 is negative.

Taking into account the fact that $\delta = 1$ in cases 1 and 3 and $\delta = -1$ in cases 2 and 4 we obtain that, on the boundary curves $\gamma_c^{(0)}$, $\gamma_c^{(2k)}$, $\gamma_s^{(2k-1)}$ $(k = 1, 2, \ldots)$ the solution $q = p = 0$ of Eqs (1.2) is unstable, while on the boundary curves $\gamma_c^{(2k-1)}$, $\gamma_s^{(2k)}$, $(k = 1, 2, \ldots)$ we have stability.

The results of Sections 3–5 show that the theorem formulated in Section 2 is correct. Together with the known results of an investigation of instability in the first approximation presented in Section 2 it gives a comprehensive answer to the question of the Lyapunov stability of the normal and inverted positions of the pendulum for vertical harmonic oscillations of its point of suspension.

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